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THE LOCALIZED SINGLE-VALUED EXTENSION PROPERTY AND LOCAL SPECTRAL THEORY

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Abstract. In this paper, we study properties SVEP, Bishop's property (β), property (δ), decomposability, property $(\beta)_\epsilon$, subscalarity, Kato spectrum, generalized Kato decomposition, finite ascent for bounded linear operators in Banach spaces.

1. Introduction and preliminaries

Throughout this paper, let X be a non-zero complex Banach space over the complex field $\mathbb{C}$ and let $L(X)$ be the Banach algebra of all bounded linear operators on X. As usual, given $T \in L(X)$, we denote by $N(T)$, $\sigma_{ap}(T)$, $\sigma(T)$ and $\rho(T)$ the kernel, the approximate point spectrum, the spectrum and the resolvent set of T, respectively and let $Lat(T)$ stand for the collection of all T-invariant closed linear subspaces of X. For a T-invariant closed linear subspace Y of X, let $T|Y$ denote the operator given by the restriction of T to Y.

The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ such that there exist an analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in U$.

The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Obviously, $\sigma_T(x)$ is a closed subset of $\sigma(T)$.
For arbitrary $T \in L(X)$ and $F \subseteq \mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *local spectral subspace* of T. It is clear that $X_T(F)$ is a hyperinvariant subspace of X, but need not closed in general, see [8] and [14].

An operator $T \in L(X)$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U centered at λ_0, the only analytic function $f : U \to X$ which satisfies the

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equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the constant function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. The identity theorem for analytic functions ensures that every $T \in L(X)$ has SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, every operator has the SVEP at every isolated point of the spectrum. Clearly, T has SVEP at λ precisely when $\lambda I - T$ has SVEP at 0. It follows from Proposition 1.2.16 [14] that T has the SVEP if and only if $X_T(\phi) = \{0\}$, and this is the case if and only if $X_T(\phi)$ is closed.

We recall that an operator $T \in L(X)$ is called *decomposable* provided that for each open cover $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in Lat(T)$ for which

$$Y + Z = X, \quad \sigma(T|Y) \subseteq U \text{ and } \sigma(T|Z) \subseteq V.$$

This class is quite general, containing for example all compact operators, normal operators on a Hilbert spaces, Dunford's spectral operators and generalized scalar operators, see the monographs by [1], [8] and [14].

The single-valued extension property dates back to the early days of local spectral theory and appeared in the work of Dunford [9], [10]. A thorough discussion of this property within the theory of spectral and generalized spectral operators may be found in the seminal monographs by Dunford-Schwartz [11] and Colojoară and Foias [8]. The localized version of single-valued extension property was introduced by Finch [12]. As witness by the more recent accounts in [1] and [14], localized single-valued extension property has now developed into one of the major tools in the local spectral theory and Fredholm theory for operators on Banach spaces.

In this paper, we proved that if $TST = T^2$ and $STS = S^2$ then T, S, ST and TS share many spectral properties and local spectral properties as subscalarity, finite ascent, semi-regularity, GKD, decomposability, Bishop's property (β), Dunford's property (C), SVEP. We also proved that if $TST = T^2$ and $STS = S^2$ then $\sigma_K(T)\setminus\{0\} = \sigma_K(S)\setminus\{0\}$.

2. Main results

We consider the case that the operators $S, T \in L(X)$ satisfy the operator equations $TST = T^2$ and $STS = S^2$. This case was studied first in [20]. It is well known [20] that if $A, B \in L(X)$ such that $A^2 = A$,

$B^2 = B$, $T = AB$ and $S = BA$, then $TST = T^2$ and $STS = S^2$. Moreover, $\omega(T) \setminus \{0\} = \omega(S) \setminus \{0\}$ for the spectrum $\omega = \sigma$, as well as for the point spectrum, approximate point spectrum, residual spectrum, continuous spectrum, Fredholm spectrum. If $S, T \in L(X)$ satisfy the operator equations $TST = T^2, STS = S^2$, and $0 \in \rho(T)$ or $0 \in \rho(S)$ then clearly $T = I = S$. So we always assume that T and S are not invertible.

For an arbitrary operator $T \in L(X)$, let

$$D(T) := \{\lambda \in \mathbb{C} \ : \ T \text{ fails to have SVEP at } \lambda\}.$$

Obviously, $D(T)$ is empty precisely when T has SVEP. Moreover, it follows readily from the identity theorem for analytic functions that $D(T)$ is open, and therefore contained in the interior of the spectrum $\sigma(T)$.

Theorem 2.1. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then $D(T) = D(ST) = D(TS) = D(S)$.*

Proof. First, we prove that $D(T) = D(ST)$. Suppose that $\lambda_0 \notin D(T)$. Then T has SVEP at λ_0. Let $f : U \to X$ be an analytic function defined on an open neighborhood U of λ_0 such that

$$(\lambda I - ST)f(\lambda) = 0 \ \text{ for all } \lambda \in U. \tag{2.1}$$

Because of $TST = T^2$, we have

$$(\lambda I - T)Tf(\lambda) = T(\lambda I - ST)f(\lambda) = 0 \ \text{ for all } \lambda \in U, \tag{2.2}$$

In equation (2.2), the SVEP of T at λ_0 entails that $Tf(\lambda) = 0$ for all $\lambda \in U$, and hence $STf(\lambda) = 0$ for all $\lambda \in U$. In equation (2.1), we deduce $f(\lambda) = 0$ for all $0 \neq \lambda \in U$. By the continuity of f, $f(\lambda) = 0$ for all $\lambda \in U$, and hence ST has SVEP at λ_0. This implies that $D(ST) \subseteq D(T)$. Conversely, we assume that $\lambda_0 \notin D(ST)$. Then ST has SVEP at λ_0. Let $g : V \to X$ be an analytic function defined on an open neighborhood V of λ_0 such that

$$(\lambda I - T)g(\lambda) = 0 \ \text{ for all } \lambda \in V. \tag{2.3}$$

Then we have

$$(\lambda T - T^2)g(\lambda) = 0 \ \text{ for all } \lambda \in V. \tag{2.4}$$

Because of $T^2 = TST$, we have

$$(\lambda I - ST)STg(\lambda) = ST(\lambda I - T)g(\lambda) = 0 \ \text{ for all } \lambda \in V, \tag{2.5}$$

In equation (2.5), the SVEP of ST at λ_0 entails that $STg(\lambda) = 0$ for all $\lambda \in V$, and hence $T^2g(\lambda) = TSTg(\lambda) = 0$ for all $\lambda \in V$. In equation

(2.4), we deduce $Tg(\lambda) = 0$ for all $0 \neq \lambda \in V$. From (2.3), we have $g(\lambda) = 0$ for all $0 \neq \lambda \in V$. By the continuity of g, $g(\lambda) = 0$ for all $\lambda \in V$, and hence T has SVEP at λ_0. This implies that $D(T) \subseteq D(ST)$. Similar arguments as above show that $D(ST) = D(TS) = D(S)$. □

Following the similar proof of Theorem 2.1, we can easily show the next corollary.

Corollary 2.2. *Let $T, S \in L(X)$. Then $D(ST) = D(TS)$. In particular, ST has SVEP if and only if TS has SVEP.*

An operator $T \in L(X)$ on a complex Banach space X has *Dunford's property (C)* if $X_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$. Evidently, Dunford's property (C) implies that SVEP.

Theorem 2.3. ([3]) *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. If one of operators T, ST, TS and S has Dunford's property (C), then all the operators satisfy Dunford's property (C).*

Let U be an open subset of the complex numbers $\mathbb{C}$ and $H(U, X)$ be the Frécht algebra of all analytic X-valued functions on U endowed with uniform convergence on compact sets of U. Recall that an operator $T \in L(X)$ is said to satisfy *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$, open disc centered at λ with radius r, and for any sequence $(f_n)_n \subset H(U, X)$ if $\lim_{n\to\infty}(\mu I - T)f_n(\mu) = 0$ in $H(U, X)$ then $\lim_{n\to\infty} f_n(\mu) = 0$ in $H(U, X)$. For an arbitrary operator $T \in L(X)$, let

$$\sigma_\beta(T) := \{\lambda \in \mathbb{C} : T \text{ fails to satisfy Bishop's property } (\beta) \text{ at } \lambda\}.$$

We say that $T \in L(X)$ satisfies *Bishop's property* (β) precisely when $\sigma_\beta(T) = \phi$. Obviously, Bishop's property (β) implies Dunford's property (C).

Theorem 2.4. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. If one of operators T, ST, TS and S has Bishop's property (β) at $\lambda \in \mathbb{C}$, then all the operators satisfy Bishop's property (β) at $\lambda \in \mathbb{C}$. Moreover,*

$$\sigma_\beta(T) = \sigma_\beta(ST) = \sigma_\beta(TS) = \sigma_\beta(S).$$

Proof. It suffices to show that if T has Bishop's property (β) at $\lambda_0 \in \mathbb{C}$ then ST, TS and S satisfies Bishop's property (β) at $\lambda_0 \in \mathbb{C}$. Let $(f_n)_n$ be a sequence of X-valued analytic functions in a neighborhood U of λ_0 such that

$$\lim_{n\to\infty}(\lambda I - ST)f_n(\lambda) = 0 \quad \text{in } H(U, X), \tag{2.6}$$

Since $T^2 = TST$, we have

$$\text{(2.7)} \quad \lim_{n\to\infty}(\lambda I - T)Tf_n(\lambda) = \lim_{n\to\infty} T(\lambda I - ST)f_n(\lambda) = 0 \text{ in } H(U,X),$$

Suppose that T has Bishop's property (β) at λ_0. By (2.7), $\lim_{n\to\infty} Tf_n(\lambda) = 0$ in $H(U,X)$ and then

$$\lim_{n\to\infty} STf_n(\lambda) = 0 \quad \text{in } H(U,X).$$

It follows from (2.6) that $(\lambda f_n(\lambda))_n$ converges to 0 on each compact set. By the maximum modulus principle, $(f_n)_n$ converges to 0 on each compact set, and hence ST has Bishop's property (β) at λ_0. Similar arguments as above show that both TS and S satisfies Bishop's property (β) at $\lambda_0 \in \mathbb{C}$. □

COROLLARY 2.5. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then the following assertions are equivalent;*

(a) *T has Bishop's property (β);*
(b) *ST has Bishop's property (β);*
(c) *TS has Bishop's property (β);*
(d) *S has Bishop's property (β).*

Recall from [14] that an operator $T \in L(X)$ is said to have the *decomposition property* (δ) if the adjoint operator T^* on the dual space X^* satisfies Bishop's property (β).

In [4], Albrecht, Eschmeier and Neumann showed that an operator $T \in L(X)$ is decomposable if and only if T has both properties (β) and (δ). Moreover, Albrecht and Eschmeier proved that the property (β) and (δ) are dual to each other in the sense that an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* on the dual space X^* satisfies (δ) and that the corresponding statement remains valid if both properties are interchanged. Obviously, an operator $T \in L(X)$ is decomposable precisely when both T and T^* have property (β).

COROLLARY 2.6. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then the following statements are equivalent;*

(a) *T has property (δ) (resp. decomposable) ;*
(b) *TS has property (δ) (resp. decomposable) ;*
(c) *ST has property (δ) (resp. decomposable) ;*
(d) *S has property (δ) (resp. decomposable).*

The *Kato resolvent set* $\rho_K(T)$ of an operator $T \in L(X)$ on a complex Banach space X is defined as the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - T)X$ is closed and $N(\lambda I - T) \subseteq (\lambda I - T)^\infty(X)$, where $(\lambda I - T)^\infty(X) :=$

$\bigcap_{n=1}^{\infty}(\lambda I - T)^n X$ is the *hyperrange* of T. The *Kato spectrum* of $T \in L(X)$ is defined by $\sigma_K(T) =: \mathbb{C} \setminus \rho_K(T)$. The Kato spectrum $\sigma_K(T)$ is always a non-empty compact subset of $\mathbb{C}$ and

$$\partial\sigma(T) \subseteq \sigma_K(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T).$$

Moreover, $\sigma_K(T) \subseteq \sigma_{ap}(T) \cap \sigma_{sur}(T)$, where $\sigma_{sur}(T) := \{\lambda \in \mathbb{C} : (\lambda I - T)(X) \neq X\}$ is the surjectivity spectrum of T. Furthermore, $\sigma_K(f(T)) = f(\sigma_K(T))$ for every analytic function f in a neighborhood of $\sigma(T)$, see [1] and [14].

Theorem 2.7. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then*

$$\sigma_K(T) \setminus \{0\} = \sigma_K(TS) \setminus \{0\} = \sigma_K(ST) \setminus \{0\} = \sigma_K(S) \setminus \{0\}.$$

Proof. Assume to the contrary that $\lambda \in \rho_K(T)$ and $\lambda \neq 0$. Then by Theorem 1.2 of [19], we have $\lambda \notin \sigma_{ap}(TS)$. Thus $N(\lambda I - TS) = \{0\}$ and $(\lambda I - TS)(X)$ is closed, which implies that $\lambda \in \rho_K(TS)$. The same arguments as above show that $\sigma_K(A) \setminus \{0\} = \sigma_K(B) \setminus \{0\}$, where $A, B = T, ST, TS, S$. □

Corollary 2.8. *Let $T, S \in L(X)$. Then $\sigma_K(TS) \setminus \{0\} = \sigma_K(ST) \setminus \{0\}$.*

Let $E(U, X)$ be the Frécht algebra of all infinitely differentiable X-valued functions on $U \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of U of all derivatives. Recall that an operator $T \in L(X)$ is said to have *property* $(\beta)_\epsilon$ at $\lambda_0 \in \mathbb{C}$ if there exists a neighborhood U of λ_0 such that for each open set $O \subset U$ and for any sequence $(f_n)_n$ of X-valued functions in $E(U, X)$ the convergence of $(\lambda I - T)f_n(\lambda)$ to zero in $E(U, X)$ yields to the convergence of f_n to zero in $E(U, X)$. For an arbitrary operator $T \in L(X)$, let

$$\sigma_{(\beta)_\epsilon}(T) := \{\lambda \in \mathbb{C} : T \text{ fails to satisfy } (\beta)_\epsilon \text{ at } \lambda\}.$$

We say that $T \in L(X)$ satisfies *property* $(\beta)_\epsilon$ if precisely when $\sigma_{(\beta)_\epsilon}(T) = \phi$. It is well known that property $(\beta)_\epsilon$ characterizes those operators with some generalized scalar extension.

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. An operator $T \in L(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$, the identity operator on X and $\Phi(z) = T$, where 1 denotes the constant function on $\mathbb{C}$ and z the identity function on $\mathbb{C}$.

A bounded linear operator is said to be *subscalar* if it is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces.

THEOREM 2.9. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. If one of operators T, ST, TS and S has property $(\beta)_\epsilon$ at $\lambda_0 \in \mathbb{C}$, then each operators satisfies property $(\beta)_\epsilon$ at $\lambda_0 \in \mathbb{C}$. Moreover,*

$$\sigma_{(\beta)_\epsilon}(T) = \sigma_{(\beta)_\epsilon}(ST) = \sigma_{(\beta)_\epsilon}(TS) = \sigma_{(\beta)_\epsilon}(S).$$

Proof. We have to show that if T has property $(\beta)_\epsilon$ at $\lambda_0 \in \mathbb{C}$ then ST, TS, and S satisfies property $(\beta)_\epsilon$ at $\lambda_0 \in \mathbb{C}$. Suppose that $\lambda_0 \notin \sigma_{(\beta)_\epsilon}(T)$. Let U be a neighborhood of λ such that $U \cap \sigma_{(\beta)_\epsilon}(T) = \phi$, and let $(f_n)_n$ be a sequence in $E(U, X)$ such that

$$\lim_{n\to\infty} (\lambda I - ST) f_n(\lambda) = 0 \ \text{ in } \ E(U, X). \tag{2.8}$$

Then we have

$$\lim_{n\to\infty} (\lambda I - T) T f_n(\lambda) = \lim_{n\to\infty} T(\lambda I - ST) f_n(\lambda) = 0 \ \text{ in } \ E(U, X).$$

Because of $\lambda_0 \notin \sigma_{(\beta)_\epsilon}(T)$, $\lim_{n\to\infty} T f_n(\lambda) = 0$ for all $\lambda \in U$ and then

$$\lim_{n\to\infty} ST f_n(\lambda) = 0 \ \text{ in } \ E(U, X).$$

From (2.8), we obtain $(\lambda f_n(\lambda))_n$ converges to 0 in $E(U, X)$. By Lemma 2.1 of [7], $(f_n(\lambda))_n$ converges to 0 in $E(U, X)$ and hence ST has property $(\beta)_\epsilon$ at $\lambda_0 \in \mathbb{C}$. This implies that $\sigma_{(\beta)_\epsilon}(ST) \subseteq \sigma_{(\beta)_\epsilon}(T)$. Similar arguments as above show that

$$\sigma_{(\beta)_\epsilon}(S) \subseteq \sigma_{(\beta)_\epsilon}(TS) \subseteq \sigma_{(\beta)_\epsilon}(ST) \subseteq \sigma_{(\beta)_\epsilon}(T).$$

It remains to show that $\sigma_{(\beta)_\epsilon}(T) \subseteq \sigma_{(\beta)_\epsilon}(S)$. It follows from arguments similar to those in the proof of Theorem 2.4, we omit the proof. □

COROLLARY 2.10. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then the following assertions are equivalent;*

(a) *T is subscalar;*
(b) *TS is subscalar;*
(c) *ST is subscalar;*
(d) *S is subscalar.*

COROLLARY 2.11. *Let $T, S \in L(X)$. If T and S are injective, then TS is subscalar if and only if ST is subscalar.*

Recall that an operator $T \in L(X)$ is said to have *finite ascent* if $N(T^p) = N(T^{p+1})$ for some positive integer p. The *ascent* of $T \in L(X)$, defined as $p(T) := inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, and *descent* of $T \in L(X)$, defined as $q(T) := inf\{n \in \mathbb{N} : T^n(X) = T^{n+1}(X)\}$, the infimum over the empty set is taken ∞. It is well known from Proposition 38.3 [13] that $p(T)$ and $q(T)$ are both finite then if $p(T) = q(T)$. Also, it is well known from [1] that the following implications hold:

$$p(\lambda I - T) < \infty \quad \Longrightarrow \quad T \text{ has SVEP at } \lambda,$$

and dually

$$q(\lambda I - T) < \infty \quad \Longrightarrow \quad T^* \text{ has SVEP at } \lambda.$$

The property that $\lambda I - T$ has finite ascent is closely connected to λ being a pole of the resolvent of T, see more details [1] and [13].

Proposition 2.12. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then the following assertions are equivalent;*

(a) *T has finite ascent ;*
(b) *TS has finite ascent ;*
(c) *ST has finite ascent ;*
(d) *S has finite ascent.*

Proof. By Proposition 10 [6] and Corollary 2.2 [21]. □

As usual, an operator $T \in L(X)$ is said to be *semi-regular* if $T(X)$ is closed and $N(T) \subseteq T^\infty(X)$. Evidently, T is semi-regular if and only if $0 \in \rho_K(T)$. It is well known that the Kato resolvent set $\rho_K(T) = \rho_K(T^*)$ is an open subset of $\mathbb{C}$. Consequently, Theorem 2.1 together with Théoréme 2.7 of [17] yields the following corollary.

Corollary 2.13. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. If T has SVEP then $\rho_K(T) = \rho_K(TS) = \rho_K(ST) = \rho_K(S)$. Moreover, the following statements are equivalent;*

(a) *T is semi-regular;*
(b) *TS is semi-regular;*
(c) *ST is semi-regular;*
(d) *S is semi-regular.*

An operator $T \in L(X)$ is said to admit *a generalized Kato decomposition*(GKD), if there exists a pair of $T-$invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$ is semi-regular and $T|N$ is quasi-nilpotent. If we assume in the definition above that $T|N$ is nilpotent, i.e. there exists $d \in \mathbb{N}$ for which $(T|N)^d = 0$, then T is said to be of *Kato type of operator of order d,* see more details [1]. An important class of operators of Kato type is given by the class of all semi-Fredholm operators. Obviously, every semi-regular operator has a GKD such as $M = X$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD such as $M = \{0\}$ and $N = X$. We denote as usual its spectral radius by $r(T) := max\{|\lambda| : \lambda \in \sigma(T)\}$.

THEOREM 2.14. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Suppose that $T \in L(X)$ and $S \in L(X)$ are surjective. If T has SVEP then the following statements are equivalent.*

(a) *T admits a GKD;*
(b) *TS admits a GKD;*
(c) *S admits a GKD;*
(d) *ST admits a GKD.*

Proof. $(a) \Rightarrow (b)$. Suppose that T admits a GKD. Then there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$ is semi-regular and $T|N$ is quasi-nilpotent. Let $Y := \overline{T^2(M)}$ and $Z := \overline{T^2(N)}$. Then, clearly Y and Z are closed TS-invariant subspaces. Since T is surjective,

$$X = T^2(M) + T^2(N) \subseteq \overline{T^2(M)} + \overline{T^2(N)} = Y + Z,$$

and then $X = Y + Z$. Since $M, N \in Lat(T)$, we have

$$Y \cap Z = \overline{T^2(M)} \cap \overline{T^2(N)} \subseteq \overline{M} \cap \overline{N} = \{0\},$$

and hence $X = Y \oplus Z$. By Corollary 2.13, TS is semi-regular. Since semi-regularity is inherited by restrictions on closed invariant subspaces, $TS|Y$ is semi-regular. Note that, since $r(TS|Z) \leq r(T|Z)r(S|Z)$ and $T|N$ is quasi-nilpotent, the product $TS|Z$ is quasi-nilpotent. Hence TS admits a GKD.

$(b) \Rightarrow (c)$. Suppose that TS admits a GKD. Let $M_1, N_1 \in Lat(TS)$ be such that $X = M_1 \oplus N_1$, the restriction $TS|M_1$ is semi-regular and $TS|N_1$ is quasi-nilpotent. Then clearly $Y_1 := \overline{S(M_1)}$ and $Z_1 := \overline{S(N_1)}$ are closed S-invariant subspaces. Since S is surjective, we have

$$X = S(M_1) + S(N_1) \subseteq \overline{S(M_1)} + \overline{S(N_1)} = Y_1 + Z_1,$$

and hence $X = Y_1 + Z_1$. Because of $M_1, N_1 \in Lat(TS)$, we have

$$Y_1 \cap Z_1 = \overline{S(M_1)} \cap \overline{S(N_1)} \subseteq \overline{M_1} \cap \overline{N_1} = \{0\},$$

and hence $X = Y_1 \oplus Z_1$. By Corollary 2.13, S is semi-regular and hence $S|Y_1$ is semi-regular. Since $TS|N_1$ is quasi-nilpotent, the product

$STS|Z_1 = S^2|Z_1$ is quasi-nilpotent and hence $S|Z_1$ is quasi-nilpotent. Therefore S admit a GKD.

$(c) \Rightarrow (d)$. In the proof of $(a) \Rightarrow (b)$, we interchange T and S.

$(d) \Rightarrow (a)$. The proof is similar to that of $(b) \Rightarrow (c)$. $\square$

COROLLARY 2.15. *Suppose that $T \in L(X)$ and $S \in L(X)$ are surjective. If TS has the SVEP then TS admits a GKD if and only if ST admits a GKD.*

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